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SOME CONDITIONS FOR COMPLETENESS IN COUNTABLE-VALUED LOGIC

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(Submitted by A. N. Kolmogorov, 14 May 1959)

Let P_m be the set of all functions, determined for the set E^m of power m and which also takes on values within this set. The closed subset $\mathfrak{M}$ and set P_m with respect to its superposition (1) is called a closed class. The subset $\mathfrak{N}$ is called the complete system in P_m if its closure with respect to its operation of superposition is coinciding with the set P_m. The subset $\mathfrak{M}$ is called pre-completed class [partially-completed], if it is not a complete system, but by adding to it any one function from P_m which does not belong to $\mathfrak{M}$, will give us a complete class within the P_m system.

In our further discussion we will be interested in sets P_m for which m = k (k is a natural number greater than one) and $m = \aleph_0$. The first is called k-valued logic (1), the second countable-valued logic (2). In case m = k then for the set E^k we can take the set $\{0, 1, \ldots, k-1\}$, anf if $m = \aleph_0$ then $E^{\aleph_0} = \{0, 1, 2, \ldots\}$.

One of the important questions in respect to k-valued logic and countable-valued logic is the question of completeness of the system of functions. For P_k the following theorem of completeness is taking place (1):

In order for the system of functions P_k to be complete, it is necessary and sufficient that it be not included in any of the pre-completed classes of P_k. As it is evident, the whole collection of pre-completed classes of P_k is considered in this theorem. The power of this collection being finite increases rather swiftly with the increase of k. Therefore by presenting itself as an effective and sufficiently good (from the theoretical point of view) criterion of completeness, the introduced theorem, nevertheless, presents considerable difficulties for the practical realization, in exposing the completeness of concrete systems when k is sufficiently large. In view of this, in k-valued logic other theorems of completeness, which are less complex from a practical point of view, take first place. To such theorems belongs, in particular, the criterion of

Slupetsky, which displays a sufficient condition for completeness (1): that those systems of functions are complete in P_k, which consist of all functions from one argument, and functions, essentially dependable on not less than two variables and taking on all k values.

One should take notice that the knowledge of the pre-completed classes gives the necessary conditions of the completeness of systems of functions; therefore the search for the new pre-completed classes presents an interest in regard to the solution of the question of completeness in P_k.

In countable-valued logic, because of the impossibility to obtain an effective criterion of completeness with the use of pre-completed classes (inasmuch as the power of the set of all the pre-completed classes of $P_{\aleph_0}$ is not less than that of continuum), and also in view of the fact that we have not yet solved the question of the sufficiency of the condition of noninclusion of the system in any one of the pre-completed classed to obtain the completeness of this system, special importance is being acquired by the theorems of completeness, that do not consider the whole aggregrate of pre-completed classes in $P_{\aleph_0}$. As in k-valued logic, locations of the new types of pre-completed classes in $P_{\aleph_0}$ allow us to obtain more and more perfect necessary conditions of completeness. Sufficient conditions of completeness can be found by way of analyzing the conditions of systems of special form, in particular, systems analogic to those investigated by Slupetsky's criterion. One should consider, that a straight generalization of Slupetsky's theorem for the counting case does not give a positive result (see theorem 3).

Until now only one family of pre-completed classes of $P_{\aleph}$ were known, so called classes which retained the set $E \subset E^{\aleph_0}$ (1), and one sufficient sign of completeness (for systems composed of all functions, of one variable, and "peanofs" function)*.

In the present notation** we cite one series (family) of pre-completed classes and one pre-completed class, and are giving two indications (signs) of the completeness of systems of a special form (Slupetsky'x type).

1. The decomposition of D of the set $E^{\aleph_0} = \{0,1,2,\ldots\}$:

$$E^{\aleph_0} \sum_{i=1}^{\infty} \mathcal{E}_i \ (\mathcal{E}_i \mathcal{E}_j = \Lambda \ \text{ if } \ i \neq j)$$

*The latter was reported on by A. V. Kuznetzov, as a consequence of a more general outcome, at the seminar of mathematical logic at the Moscow State University in 1951.

**The present notation is basically a short presentation of my diploma, written at the Moscow State University in 1958 under the guidance of S. V. Yablonskiy.

is called correct, if it satisfied the following requirements: 1)the power of the subset $\mathscr{E}_i$ is finite for any one i; 2) the number of subset $\mathscr{E}_i$, for which the power is greater than one unit, is finite 3) if there exists at least one subset of decomposition with power greater than one unit.

Function $f(x_1, x_2, \dots, x_n)$ is called the function that preserves the decomposition, (in particular the proper decomposition) D, in case that the collections $\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ of values of the arguments are equivalent $\tilde{\alpha} \sim \tilde{\beta}$ (mod D), i.e. α_i and $\beta_i \in \mathscr{E}_j$ at $i = 1, 2, \dots, n$, it follows that also the values of these collections are equivalent:

$f(\tilde{\alpha}) \sim f(\tilde{\beta})$ (mod D).

We will call the set of all the functions containing the proper decomposition D of the set $E^{\aleph_0}$, the class of type W and we will denote it by W(D).

Theorem 1. The class of type W is a pre-completed class.

Obviously, the maximum power of a set of pairs of nondual classes of the considered type is the cardinal number $\aleph_0$.

The set $U_s(g(x)) = \{t : t \in E^{\aleph_0} \,\&\, g(t) = s\}$ where $s \in E^{\aleph_0}$ and the function g(x) belong to the set G (everywhere in this notation G is the set of function of $P_{\aleph_0}$, dependent on not more than one argument), is called the interval of continuity of the function g(x). The function g(x) will be called definite-continuous, if it has no more than a finite number of counting-power intervals of continuity. The set of all finitely-continuous functions of $P_{\aleph_0}$, we will denote by Q. The class of of all the functions of $P_{\aleph_0}$, preserving the set Q (i.e. the class which consists of all the functions $f(x_1, x_2, \dots, x_n)$, $n \geqslant 0$ such that the function $f(g_1(x), g_2(x), \dots, g_n(x)) \in Q$ at any $g_i(x) \in Q$, $i = 1, 2, \dots, n$) we will call the class V. It is easy to see that $Q \subset V$

Theorem 2. The class V is pre-completed.

2. For further discussion, a series of conceptions will be necessary, and the introduction of them will allow, in a compact and also convinient for a survey form, to give two indications of the completeness of systems which are similar to the systems considered by Slupetsky's criterion in P_k.

The function $f(x_1, x_2, \dots, x_n) \in P_{\aleph_0}$ ($n \geqslant 2$ and all the existing variables) is called "peanof's", if for different collections of values of the arguments it takes on different values. An example of peonof's function of two arguments is given in table 1. The function $f(x_1, x_2, \dots, x_n)$ is called quasipeanof's,

if the system $G_f = G \cup \{f\}$ is complete in $P_{\aleph_0}$.

We will analyze the set G_f, where f is a function, substantially dependent on two arguments. To each superposition of the function of the set G_f, that represents thefunction dependable of not more than on two arguments, we will put in correspondence a certain number, defined by the following rule: 1) for x (or y) $\nu(x)=0$ $(\nu(y)=0)$ 2) for $g(\varphi(x,y)) - \nu(g(\varphi(x,y)))=\nu(\varphi(x,y))$ 3) for $f(\phi_1(x,y),\phi_2(x,y)) - \nu(f(\phi_1(x,y),\phi_2(x,y))) = \max(\nu(\phi_1(x,y)),$ $\nu(\phi_2(x,y)))+1$ $(\varphi(x,y), \phi_1(x,y)$ and $\phi_2(x,y)$ — is the superposition of the functions of the set G_f, which represent functions depending on not more than two arguments).

Table 1

$y \backslash x$	0	1	2	3	...
0	0	1	3	6	...
1	2	4	7	11	...
2	5	8	12	17	...
3	9	13	18	24	...
.	.	.	.	.	
.	.	.	.	.	
.	.	.	.	.	

Table 2

$y \backslash x$	0	1	2	3	...
0	0	2	4	6	...
1	1	0	1	2	...
2	3	0	0	3	...
3	5	0	0	0	...
.	.	.	.	.	
.	.	.	.	.	
.	.	.	.	.	

Let F(x,y) be the superposition of the functions of the system $G_f(f=f(x,y))$. We will call it the superposition of the r's order with respect to f, if $\nu(F(x,y))=r$. The definition given is generalized rather easily in case of the number of arguments being greater than two.

The function f (x, y) is called quasipeanof's function of r's order, if there exists a superposition of r-order of the functions of the system G_f such that the function represented by it

(of two arguments) is a peanof's function, but there is no superposition of a lesser order with the same property.

We will analyze the system of H_f functions, produced by the function $f(x_1, x_2, \ldots, x_n)$ $(n \geqslant 2$ and for all the existing variables) and by the functions of the set G: $H_f = \{\varphi : \varphi(x_{j_1}, x_{j_2}, \ldots, x_{j_m}) = g_0(f(g_1(x_{i_1}), g_2(x_{i_2}), \ldots, g_n(x_{i_n}))), g_l(x) \in G\ (l = 0, 1, \ldots, n), 1 \leqslant i_q \leqslant n, m \geqslant 0\}$ Obviously, if some function of H_f is quasipeanof's function of r order, then $f(x_1, x_2, \ldots, x_n)$ is a quasipeanof's function of the order not greater than r. It is easy to notice, that for a function to be a quasipeanof function of the 1st order, it is necessary and sufficient that its system of H_f contain a peanof function.

The function f(x,y) is called degenerated, if f(k,y) is a function with a finite number of intervals of continuity at any $k \in E^{\aleph_0}$ or f(x,l) is a function with the same property at any $l \in E^{\aleph_0}$. Otherwise the function f(x,y) is called non-degenerate.

Theorem 3. For a function f(x,y) to be a quasipeanof's function, it is necessary that it be nondegenerate.

The proof is carried out by induction of the order of superposition with respect to function f.

We will call the function f(x,y) diagonally-divided, if it possesses the following properties: 1) its H-system does not include a peanof function; 2a) the h-system includes the function $\varphi(x, y)$, for which the set $\Phi_0 = \varphi(\{0\}, E^{\aleph_0} \setminus \{0\})$ consists of different numbers, the set $\Phi_1 = \varphi(E^{\aleph_0} \setminus \{0\}, \{0\})$ consists of different numbers, the set $\Phi_2 = \varphi(\{0\}, \{0\}) \bigcup \varphi(\{x : x > 0\}, \{y : y \geqslant x\}) = \{0\}$ the set $\Phi_3 = \varphi(\{x : x > y\}, \{y : y > 0\})$ consists of different numbers, where $\Phi_0 \Phi_1 = \Phi_0 \Phi_2 = \Phi_1 \Phi_2 = \Phi_2 \Phi_3 = \Lambda$ or 26) the H-system includes the function $\varphi(x, y)$ for which the set $\Phi_1 = \varphi(E^{\aleph_0} \setminus \{0\}, \{y : y \leqslant x\})$ consists of different numbers, the set $\Phi_2 = \varphi(E^{\aleph_0}, \{y : y > x\})$ consists of parts (proportions) $\Phi_2 l = \varphi(\{x : x < l\}, \{l\} = a_l\ (a_l \in E^{\aleph_0}), \Phi_{2l_1} \Phi_{l_2} = \Lambda$ at $l_1 \neq l_2$, where $\Phi_1 \Phi_2 = \Lambda$.

An example of a diagonally-divided function is given in table 2. Theorem 4. A diagonally-divided function is a quasipeanof's function of the second order.

For the diagonally-divided function introduced in this example we have $f(f(2x+2, 2y+3), f(2y+3, 2x+2))$ -- a peanof's function.

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